SOME GLIMPSES OF ANCIENT HINDU MATHEMATICS.

BY A. A. KRISHNASWAMY AYYANGAR.

In ancient India, Mathematics was first developed as a Vedanga or a limb of the Vedas, an art subserving a religious purpose such as the construction of sacrificial altars. With rare intuitive ability the early Hindus grasped quickly the fundamentals of the subject and embodied them in short verses. To them, the world owes some of the basic ideas in mathematics—the place-value system of notation in Arithmetic, the generalizations of Algebra, the sine-function in Trigonometry, and the foundations of Indeterminate Analysis. India may claim to be the birth-place of Algebra and Arithmetic.

To a casual student of ancient Hindu Mathematics, the presentation may appear fragmentary and incoherent. The early texts contain merely rules, results and, sometimes, a number of problems but rarely a fully worked out mathematical argument. The Indian tradition was to make the text as brief as possible, so that the whole might be easily learned by heart thus enabling the ancient scholars to escape from the slavery of books.

The mathematical works of the early Hindus have a remarkable parallel in Ramanujan's note-books and methods of work. Dr. G. N. Watson remarks* that his (Ramanujan's) procedure in his note-books is in accordance with the traditional custom, for the early Hindus were not in the habit of preserving their proofs, though they doubtless reasoned out most or all of their discoveries. The following words of Dr. Littlewood,† about Ramanujan apply equally to the ancient Hindu mathematicians: 'Ramanujan's intuition worked in analogies sometimes very remote, and to an astonishing extent by empirical induction from particular numerical cases. He was not interested in rigour which, for that matter, is of secondary importance in analysis and can be supplied, given the real idea, by any competent professional. The

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* Lecture delivered on 5-2-1931 on 'Ramanujan's Note-books': reprinted in the *Journal of the London Mathematical Society*, Vol. VI. p. 2

† Review of the *Collected Papers of Srinivasa Ramanujan* by Littlewood

clear-cut idea of what is meant by a proof, he perhaps does not possess at all. If a significant piece of reasoning occurred somewhere and the total mixture of evidence and intuition gave him certainty, he looked no further.'

The earliest Mathematical writings.

Let us start by taking a glimpse of the Vedanga Jyotisha, avowedly the oldest Indian work bearing on astronomy. The text is very corrupt and obscure in spite of Weber's* efforts at elucidation. The style is characterised by an enigmatical brevity, strange archaisms and total want of connection between consecutive verses. It expounds the doctrine of a cycle or Yuga of five years beginning with the winter solstice and the new-moon of Magha. According to this text, a year contains 366 days, and a Yuga contains 61 Sayana months, 66 lunar months, or 67 nakshatra months. Rules are given for calculating the relative lengths of day and night for any nycthemeron. The lengths of the longest and shortest days, according to the Jyotisha, are 18 and 12 muhurtas respectively, a nycthemeron comprising 30 muhurtas. The Jyotisha mentions the positions of the moon at different parts of the year and the time of day when a parvan (or a semi-lunar month) is finished. It is not known what additional information the unintelligible portion of the work may contain. Anyhow it is evident from the intelligible calculations given therein that the early Hindus were well-acquainted with arithmetical manipulations, including fractions.

Hindu Contributions to Arithmetic.

The next earliest mathematical document of the Hindus is probably the Sulva-sutras (not later than 200 A.D.) or the rules of measurement by cords, which deal with the construction of sacrificial altars. This work naturally reminds one of the Harpedonaptæ or the rope stretchers of Egypt. As it has a purely religious purpose in view, it gives bare directions for the construction of squares, rectangles, triangles and parallelograms with given specifications. Among other things it contains:—

(1) The first arithmetical solution of the indeterminate equation $x^2 + y^2 = z^2$ in the form

$$(mn)^2 + \left(\frac{m^2 - n^2}{2}\right)^2 = \left(\frac{m^2 + n^2}{2}\right)^2$$

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* 'Uber den Vedakalendar, namens Jyotisham' by A. Weber in the *Transactions of the Royal Academy of Sciences*, Berlin, 1862.

and (2) the rational approximations corresponding to

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34}$$

$$\frac{\sqrt{\pi}}{2} = 1 - \frac{1}{8} + \frac{1}{8.29} - \frac{1}{8.29.6} + \frac{1}{8.29.6.8}.$$

According to G. R. Kaye,* this implies a knowledge of the process of converting a fraction into partial fractions with unit numerators, a knowledge most certainly not possessed by the composers of the Sulva-sutras. We shall see presently how the premise is wrong and the conclusion is irrelevant.

In the year 1881 A.D., a work on Arithmetic was discovered by a peasant near a village called Bakshali in the North-West Frontier of India. It contains a second approximation to the square-root in the form

$$\sqrt{a^2 + r} = a + \frac{r}{2a} - \frac{(r/2a)^2}{2(a + r/2a)}.\dagger$$

Incidentally we may note that this rule also finds a place in an Arab work of the 12th century.‡ The result is easily deduced by applying, twice, the usual rule for the first approximation or the usual square-root rule found in all the Indian Arithmetical works at least from Aryabhata onwards, a rule which is independent of the kind of notation used, positional or non-positional. Brahmagupta gives an application of the square-root rule to a non-positional case.§ It is not improbable therefore that the Sulva-sutra approximations were also obtained by using the

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* *Indian Mathematics*, by G. R. Kaye, p. 7.

† The Bakshali manuscript, by G. R. Kaye: *Archælogical Survey of India*, New Imperial Series, Vol. XLII, Pts. I and II.

‡ P. 254 foot-note in Smith's *History of Mathematics*, Vol. II.

§ Verses 64, 65, p. 213, *Brahmasiddhanta* of Brahmagupta, edited by Sudhakara Dvivedi, Benares, 1902.

square-root rule thus:

$$\begin{array}{rl|l}
 & 2 & 1+\frac{1}{3}+\frac{1}{3.4}-\frac{1}{3.4.34} \\
 & 1 & \\ \hline
 & 1 & \\
2+\frac{1}{3} & 7/9 & \\ \hline
 & 2/9 & \\
2+\frac{2}{3}+\frac{1}{3.4} & 2/9+\frac{1}{9.16} & \\ \hline
 & -\frac{1}{9.16} & \\
2+\frac{2}{3}+\frac{2}{3.4}-\frac{1}{3.4.34} & -\frac{1}{9.16}+\frac{1}{3^2.4^2.34^2} & \\ \hline
 & \ldots\ldots\ldots\ldots &
\end{array}$$

As Arithmetic was the *forté* of the Hindus, all geometrical problems were first analysed as problems in Arithmetic and then expressed in geometrical language. Many geometrical theorems such as Brahmagupta's well-known formula for the area of a cyclic quadrilateral were apparently discovered arithmetically. Indeed what is now known as the Pythagorean theorem was recognised by the early Hindus as a property* of two square numbers (say a^2, b^2) being together equal to a third square (c^2) and when cords of those lengths (a, b, c) were put together (accidentally or otherwise) to form a quadrilateral with a diagonal equal to c and adjacent sides a, b, a rectangle was the result revealing thus the existence of a right-angle. Thus a sort of arithmetical intuition was at the back of a geometrical discovery. The phraseology of the Sulva-sutras betrays this:

The diagonal of a rectangle produces a square equal to the sum of the squares produced by the longer and the shorter sides respectively.

The proof of this theorem, as noted by Bhaskara (1150 A.D.) is based on the property of similar triangles assumed as an axiom.†

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* Several such sets of squares are given in the Sulva-sutras;

$3^2+4^2=5^2$; $\quad 5^2+12^2=13^2$

$7^2+24^2=25^2$; $\quad 8^2+15^2=17^2$

$12^2+16^2=20^2$; $\quad 12^2+35^2=37^2$.

† For an exposition of Bhaskara's proof, *vide* p. 499. '*New Geometry*' by the present writer.

The age of Aryabhata.

Taking a leap over a few centuries from the period of the Sulva-sutras, we come to the age of Aryabhata, Varahamihira, Brahmagupta, and the unknown author of the *Suryasiddhanta* (400-650 A.D.)

According to Thibaut, Aryabhata is unaccountably placed by tradition at the head of the Indian mathematicians. To my mind he appears to be the last of the earlier school. His style is reminiscent of the Sutra period. It is condensed to the point of unintelligibility, and coldly business-like. But the importance of the work lies in the fact that it appears to be the first to give a form and an individuality to the scattered bits of mathematical knowledge that existed before his time. The role of the *Aryabhatiyam* in giving a definite bias to Indian mathematics has its historical parallel in two other great ancient mathematical compositions—the *Elements* of Euclid and the *Arithmetica* of Diophantus

Aryabhata's work attracted much attention even outside India. His treatise on Algebra was translated into Latin by G. de Lunis, an Italian mathematician of the 13th century, and there is a manuscript copy of the translation in the Bibliotheque Nationale de Florence.* In 1874, Dr. Kern brought out the first edition of the text of the *Aryabhatiyam* with the long commentary of Paramadiswara. In 1879 Rodet gave a French translation of the *Ganita-pada*, the mathematical portion of the text, with very valuable and interesting notes. Thibaut, in 1899, gave a summary of the literature about Aryabhata. Another translation of Aryabhata's work with critical notes was published in 1930, by Walter Eugene Clark of Harvard University.† All this attention was well-deserved as he was a pioneer in several respects

He was an innovator in astronomy and paid dearly for it at the hands of Brahmagupta, who heaped unreasonable abuses on him. He was also the first in India, to hint the daily rotation of the earth on its axis

In his work, again, we find for the first and the last time a new alphabetic notation adopting both the consonants and vowels to represent

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* A reference to this is given in *L'Intermediare-des-mathematicians*, July and August, 1919.

† Vide *The Aryabhatiyam of Aryabhata* by Walter Eugene Clark (1930) The Unversity of Chicago, Illinois.

even large numbers in a very concise way,

e.g. ङिशिबुण्लख्षृ = 1582237500.

Probably this notation marks an important stage in the development of the numeral notation and is a precursor of the Indian decimal notation. It suggests a method of using the same symbol, say क, with such variations as कि, कु, etc., to denote a power of a hundred multiplying the intrinsic value of the consonant. In other words, the vowels are suggestive of the local value and the consonants of the intrinsic value in this scheme of notation. To proceed from this stage to the place-value stage it required only to drop the vowels and make the position itself indicate which power of ten was intended. This would require setting apart a symbol for zero. But it took many long centuries to recognise that the zero was also a numeral on a par with other numerals, and that a separate symbol was necessary to denote it. It was only in the eleventh century, after the decimal notation with its place-value and zero had become definitely established that another genius arose to re-adopt the alphabet to the new notation. In this connection it must be observed that the alphabetic notation in India was felt more or less as a necessity owing to the exigencies of condensed metrical composition and therefore there is a greater likelihood of its being indigenous to India than a loan from Greece or elsewhere.

Aryabhata's notation was never popular in India. Even his admirer Lalla abandoned it in favour of the more popular and the more ancient word-numerals. These could more easily be remembered and could secure better metrical euphony. The first glimpses of them are found in the Vedanga-Jyotisha. The anonymous author of *Surya-siddhanta* (not later than 5th century A.D.), and Varahamihira freely use these numeral words. But Brahmagupta of the 6th century A.D. was the first to explicitly enunciate this system in the following words :

"If you want to write one, express it by everything which is unique as the earth, the moon ; two by everything which is double, as for example black and white ; three by everything which is threefold ; the nought by heaven, the twelve by the names of the Sun."

Does not this bring us very near the modern definition of number, for, according to Bertrand Russell, 'a number is any collection which is the number of one of its members, or more simply still, a number is anything which is the number of some class'?

Side by side with the popular word-numeration there was the equally popular Brahmi notation, used generally in inscriptions, and the two combined together to produce the modern system of numerals with the place-value and zero.*

The formulation of puzzle problems and their solution proved to be a new incentive to mathematical research in Ancient India. Regarding his miscellaneous problems Brahmagupta remarks: 'These questions are stated merely for pleasure. The proficient may devise a thousand others or may solve the problems of others by the rules given here. As the stars are obscured by the sun, so does the expert eclipse the glory of other astronomers in an assembly of people by proposing algebraic problems and still more by their solution.' The later mathematical works contain many puzzle problems mainly based on indeterminate analysis. To quote one instance, Bhaskara cites the following as due to earlier authors:

"Three traders with 6, 8, and 100 panas as capital respectively having bought fruits at a uniform rate, sold a part in lots and disposed of the remainder at one for 5 panas and thus became equally rich. What were the purchase and the sale-prices?"†

The rule for solving such problems is enunciated by Vishnugupta, Mahaviracharya and others thus:

"The required sale-price is any number greater than the greatest of the given capitals, while the purchase-price is less than the product of the two sale-prices by unity."

It is interesting to note that the puzzle problems of Widmont and Blasius discussed by Glaisher‡ are perhaps the European recensions of the above Indian problem.

We give below a list of the important mathematical topics on which there are contributions from the ancient and the medieval Hindus:

(1) Surds, square-root of binomial and polynomial quadratic surds, Involution and Evolution.

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* For a detailed exposition of the growth of the so-called Hindu-Arabic numerals, *vide* an article under that name by the present writer in the *Quarterly Journal of the Mythic Society*, Vol. XVIII, No. 4, Vol. XIX, Nos. 1 and 2.

† *Jour. Ind. Math. Soc.*, Aug. 1914: 'A Classical Indian Puzzle Problem.'

‡ *Messenger of Mathematics*, Vol. LIII, No. 2.

(2) The Arithmetical and Geometrical Progressions and allied series.

(3) Simple and quadratic equations. (real roots) simultaneous equations in many variables. Equations reducible to quadratics.

Identities. Solutions of equations by means of identities.

Euler's Identity:

$$a_1 + a_2(1 + a_1) + \dots + a_n(1 + a_1)\dots(1 + a_{n-1}) = \prod_1^n (1 + a_r).$$

(4) Permutations and Combinations.

(5) Indeterminate equations of the first and the second degree.

(6) Numerical approximations for $\sqrt{n}$ and π. Approximations for a side of a regular polygon inscribed in a circle.

(7) Interpolation formulæ, especially in connection with the sine-table.

(8) Properties of triangles, quadrilaterals, circles,

(9) Mensuration of the sphere, the cone, and the pyramid.

(10) Rule of three terms (direct and inverse), rule of five, seven, nine and eleven terms; Chain-rule; Interest problems; Barter, Mixture; Motion; problems and relative velocity.

(11) Rule of Inverse operations (व्यस्तविधि).

(12) Values of π and the sine-table. Trigonometrical identities. Fundamental formulæ in spherical trigonometry.

(13) The principle of iteration (असकृत्कर्म) or successive approximations in solving some transcendental equations in astronomy. There is some analogy between this and Horner's method.

(14) The sun-dial and shadow problems. Eclipse problems based on a property of the circle.

(15) Magic squares.*

(16) Astronomy.

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* Methods of construction of magic squares appear to have been known in India even before the Christian era. A casual reference to this subject is given in Ganesa's gloss of Bhaskara's *Lilavati*, which is as late as 1520 A.D. Bhadraganita (mathematics of charms) or the construction of magic squares appears to have found a regular place in the Arithmetical treatises of Narayana and others, The Jhansi magic square is well-known (*Jour. Ind. Math. Soc.*, Feb. 1918.)

Hindu Contributions to Indeterminate Analysis.

Among the contributions of the Ancient Hindus to higher mathematics, their work on Indeterminate Analysis is the most outstanding.

Aryabhata was the first to indicate a method of getting integral solutions for the indeterminate equation of the first degree in the form,

$$mx + a = ny + b$$

(m, n, a, b being integers). Aryabhata proceeds by successive reductions* to simpler equations, until one is reached whose solution may be immediately guessed.† This is the true significance of the *mati* in Aryabhata's text and has been misunderstood by almost all the commentators, ancient and modern. Evidently they are confused between the logical and the psychological orders of evolution of mathematical ideas. Certainly the germ of the continued fraction was there in Aryabhata's method but was not clearly recognised by him It is a consolation to find that Bramagupta, who is no friend of Aryabhata, has thoroughly understood his adversary's rule and given a clearer exposition of the same. The later writers, Bhaskara and others, followed with improvements, alterations and extensions. This is, of course, to be expected. Viewed in this light, the evolution of linear indeterminate analysis from Aryabhata to Bhaskara and the later Aryabhata is quite natural.

Regarding indeterminate equations of the second degree of the type, $x^2 - Ny^2 = 1$ (N being a non-square integer), Brahmagupta

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* To illustrate: Take the equation $29x+15 = 45y+19$; *i.e.*, $x = y + \frac{16y+4}{29}$ $= y+y_1$ (say) so that $y = y_1 + \frac{13y_1 - 4}{16}$. This is the second indeterminate equation. Next set $13y_1 - 4 = 16y_2$, so that $y_1 = y_2 + \frac{3y_2 + 4}{13}$. This is the third equation. At this stage, we notice that $y_2 = 3$ obviously is a solution, and '3' is Aryabhata's *mati*.

Hence, we proceed to find the values of y_1, y, and x in order by the "rule of inverse operations" set down in all Indian mathematical works and also in the *Aryabhatiyam* in particular. The successive steps of the reverse process are indicated by Aryabhata's rule, which, in the hands of later writers, is termed *Vallika Kuttakara*.

† *Vide* pp. 18—21, *Quarterly Journal of the Mythic Society*, Jan., 1926.

(*b.* 598 A.D.) was the first to enunciate* the principle of composition of roots which is tantamount to the modern principle of composition of quadratic forms.

"If $x = a$, $y = b$ be a pair of roots of $\alpha x^2 + p = y^2$, and $x = c$, $y = d$ be roots of $\alpha x^2 + q = y^2$, then $x = bc \pm ad$, $y = bd \pm \alpha ac$ is a pair of roots of $\alpha x^2 + pq = y^2$."

The following special identities are derived by Brahmagupta from the above principle.

If $x = p$, $y = q$ is a solution in integers of $Dy^2 + 4 = x^2$, then

$$x = \frac{p(p^2 - 3)}{2}, \quad y = \frac{q(p^2 - 1)}{2}$$

is a solution in integers of $\quad Dy^2 + 1 = x^2$.

Again, if $x = p$, $y = q$ is a solution in integers of $Dy^2 - 4 = x^2$, then

$$x = (p^2 + 2)\left\{\frac{(p^2 + 3)(p^2 + 1)}{2}\right\}, \quad y = \frac{pq(p^2 + 1)}{2}\frac{(p^2 + 3)}{}$$

is an integral solution of $Dy^2 + 1 = x^2$.

His actual words are worth quoting, especially as they show how words may be abbreviated so as to possess the power of symbols in a mathematical presentation.

चतुरतिकेऽन्त्यपदकृति स्त्र्यूनादलितान्त्यपदगुणान्त्यपद ।
मन्त्यपद कृतिर्व्येकाद्विहृताद्यपदाहताद्यपदम् ॥
चतुरूनेऽन्त्यपदकृती त्र्येकयुतेवधदलं पृथग्व्येकं ।
व्येकाद्याहतमन्त्यं पदवधगुणमाद्यमान्त्यपदम् ॥

About 1150 A.D. Bhaskara surprises us by his brilliant *Chakravala* or the cyclic method, which is outlined below.

Let the roots of the equation

$$Ny^2 + k = x^2 \qquad \dots \quad (1)$$

be $x = a$, $y = b$ (N being a positive non-square integer). Solve the linear indeterminate equation $\frac{bx + a}{k} = y$ in integers, and choose a value

* In this remarkable principle Brahmagupta has anticipated Euler (1707—1783) by a thousand years. (Vide '*Diophantus of Alexandria*' by Sir T. L. Heath, p. 290)

l for x so that $|l^2 - N|$ may be the least possible. Then, the corresponding value for y is $\frac{bl + a}{k}$, which is the integral value for y in the new equation

$$Ny^2 + \frac{l^2 - N}{k} = x^2, \qquad \ldots \quad (2)$$

the corresponding integral value for x being $(al + Nb)/k$.

Just as we derived (2) from (1), we may derive another equation from (2) with known roots and so on, until the additive ultimately reduces to ± 4, ± 2, or ± 1. In case the additives are ± 4, ± 2, or -1, the roots of the equation with unity as the additive can be obtained by the principle of composition.

The true nature and importance of the *Chakravala* has not been realised even by the eminent orientalist, Colebrooke. Relying on him, probably, H. J. S. Smith misses the essential point in the cyclic method and remarks that Bhaskara and Brahmagupta misunderstood the nature of the problem.*

I have shown† that Bhaskara's method is more in line with Gauss' solution based on the theory of quadratic forms than Lagrange's, that it leads to a new set of reduced forms which I call Bhaskara forms,‡ and has the advantage of reducing the number of recurring elements and thus shortening computation

Remarkably enough, one of Bhaskara's worked examples, *viz.*, $61y^2 + 1 = x^2$, happens to be one of those proposed by Fermat in his letter to Frenicle in February 1657.

Indeed the cyclic method deserves Hankel's unstinted praise that it is certainly the finest thing achieved in the Theory of Numbers before Lagrange. Well might Colebrooke remark: 'Had an earlier translation of Hindu mathematical treatises been made and given to the public,

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* *Collected Papers*, Vol. I, pp. 192—200.

† 'New Light on Bhaskara's *Chakravala*, etc.' by the present writer in *J.I.M.S.*, Vol. XVIII (1930).

‡ A Bhaskara form is a quadratic form (A, B, C) of positive determinant N, such that

$$A^2 + C^2/4 < N, \text{ and } C^2 + A^2/4 < N.$$

especially to the early mathematicians in Europe, the progress of mathematics would have been much more rapid, since algebraic symbolism would have reached its perfection long before the days of Descartes, Pascal, and Newton;' and, I may add, Euler and Lagrange would have had a better start in their work on continued fractions.

In deference to Brahmagupta, the so-called Pellian Equation $x^2 - Ny^2 = 1$ ought to be called the Brahmagupta equation.

In passing, I may mention that a curious little problem on the indeterminate equations

$$x + a = y^2, \quad x - b = z^2$$

persists in several Indian mathematical works and appears to have gone to Europe and thence to the New World in 1556. It occurs in the American work '*Sumario Compendioso*' of one Juan Diez* in the form

'Give me a number which increased by fifteen is a square number and decreased by four is also a square number.'

I may conclude this topic by giving a sample of an apparently complex problem in indeterminate equations solved by Bhaskara.

"Tell me quickly, mathematician, two numbers such that the cube-root of half the sum of their product and the smaller number, and the square root of the sum of their squares and those square-roots) extracted from the sum and difference increased by two, and that extracted from the difference of their squares added to eight, being all five added together may yield a square-root—excepting however 6 and 8."

Contributions to Trigonometry.

Trigonometry, in its initial stages, developed as a handmaid to Astronomy in the hands of the ancient Hindus and Greeks, and this explains the rather startling fact that Spherical Trigonometry was developed earlier than Plane Trigonometry. The familiar bow and arrow must have suggested to the speculative Indian mind their mathematical analogues. The semi-chord and the arrow gave rise to the sine and the versed-sine functions. It is interesting to note that the

* Smith's *History of Mathematics*, Vol. I, pp. 355, 356.

semi-chord is an entirely Indian idea, for, even Claudius Ptolemaeus, the most important figure in the history of Greek Astronomy, gave formulae only for the full chords. While the Greeks divided the radius sexagesimally, the Hindus divided the circumference of a circle into 21600 minutes and thus straight away hit upon the circular measure.*

The early Hindus readily recognised the approximation $\sin\theta = \theta$ for small angles. They established the fundamental formulae corresponding to $\sin^2 \alpha + \cos^2 \alpha = 1$, $4 \sin^2 \alpha/2 = \sin^2 \alpha + \text{vers}^2\, \alpha$, $\text{vers}\, \alpha = 2 \sin^2 \alpha/2$ and computed relations between the half-chords and their arcs. They managed all their astronomical formulae very ingeniously with the help of the three trigonometrical functions: the sine (ज्या)†, the versed-sine (उत्क्रमज्या) and the cosine (कोटिज्या). The fundamental formulae of spherical trigonometry corresponding to

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C};$$

$$\cos A \sin c = \cos a \sin b - \sin a \cos b \cos C;$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

were also deduced most simply from the properties of right-angled triangles and the principle of similarity.

It is remarkable that almost every important Hindu astronomical work from the *Suryasiddhanta* to the *Siddhanta Siromani* gives an interpolation (or extrapolation) formula for the tabulation of successive sines. In the *Aryabhatiyam* and the *Suryasiddhanta*, we find the extrapolation formula:‡

$$\sin(n+1)\alpha - \sin n\alpha = \sin n\alpha - \sin(n-1)\alpha - \frac{\sin n\alpha}{225} \quad \text{(approx.)}$$

where n is an integer < 24 and $\alpha = 3\frac{3}{4}^\circ$.

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* It is necessary to note that the Hindu sine is a length and not a ratio like the modern sine; it is expressed, however, in terms of the arc. The early Hindus used the arc in place of the angle, attributing to it all the properties of the angle.

† The modern 'sine' is derived from the Sanskrit 'ज्या' or 'जीव.' The Arabs adapted Sanskrit 'जीव' into their Arabic 'dschiba' which they naturalised as 'dschaib' meaning 'bosom.' Later this was translated by Plato of Trivoli into 'Sinus' whence the modern 'sine.'

‡ 'The Hindu Sine-Table' by the present writer in *J.I.M S.*, Vol. XV, Dec. 1923.

In the Ganitadhyaya of *Siddhanta Siromani* Bhaskara gives an approximate quadratic interpolation formula which, in modern notation, runs

$$\sin(x-h\theta)=\sin x-\frac{\sin(x+h)-\sin(x-h)}{2}\theta$$
$$+\frac{\sin(x+h)-2\sin x+\sin(x-h)}{2}\theta^2 \qquad (\theta\leqslant 1).$$

Brahmagupta also appears to mention the above formula in his '*Khanda Khadyaka*.'*

It is worthy of note that the terms on the right-hand side of the above approximation correspond to the first three terms in the well-known general interpolation formula due to Newton.†

Hindu Astronomy.

We shall now take a rapid glance at Hindu astronomy—the science to which all the rest of mathematics was regarded as merely auxiliary.

In the most ancient Vedic hymns, we have evidences of description of the well-known astronomical phenomena, the motions of the Sun and the Moon, the seasons, the number of days in a year, the two ayanas, Devayana and Pitriyana, the intercalary month, Rita or the zodiacal belt which is the eternal track of the luminaries and the bright planets बृहस्पति and वेन. The ancient scriptures contain references (though in legendary garb) to the zodiacal shifts ‡ of the year beginning at the vernal equinox. There are sufficient data in them for postulating the phenomenon of precession of the equinoxes and determining its rate. So it is no wonder that the Siddhantas give such accurate rates as 54″ in the *Surya-siddhanta* and 59″.9007 in Bhaskara's work where the author quotes his result from Munjala. The European critics, who have a poor opinion of the ancient Hindu observations, consider this

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* 'Brahmagupta on Interpolation' by P. C. Sengupta in the *Bulletin of the Calcutta Mathematical Society*, Vol. XXIII, No. 3, 1931.

† I discussed the relation betwen this formula and Newton's corresponding Indian one in a paper submitted to the 12th Indian Science Congress, Benares, 1925. Vide *Abstracts*, p. 74.

Also Ball: *Spherical Astronomy*, p. 18, where a quadratic interpolation formula is given from which Bhaskara's result may be readily deduced.

‡ Tilak's scholarly work '*Orion*' may be consulted in this connection.

a lucky hit*, for the Greeks in spite of their (supposed) systematic observations and their marvellous inductive and deductive methods were content with a very poor precessional rate of 36″ which they probably got from the Babylonians. It has now been discovered that the precession of the equinoxes was known to the Babylonian astronomer Kidinnu long before the time of Hipparchus.

From the Vedas, we pass on to the Siddhantas. At one time there appear to have been as many as twenty†, viz., *Brahma, Surya, Soma, Brihaspati, Garga, Narada, Parasara, Poulastya, Vasishta, Vyasa, Atri, Kasyapa, Marichi, Manu, Angirasa, Lomasha, Pulisha, Yavana, Bhrigu,* and *Chyavana.* These were perhaps the earliest efforts (barring *Vedangajyotisha*) at crystallising the extant astronomical knowledge in as scientific a form as possible, about 2000 years ago. Of these the *Brahma* and the *Surya* siddhantas are the most popular and have been revised by various siddhantic writers, starting from Aryabhata.

The Siddhantas mention the number of revolutions of the sun, moon, and planets and their nodes for a Mahayuga (432×10^4 years) or a Kalpa (432×10^7 years). These huge numbers entering into Hindu calculations were necessary for accurate computation even as the seven-figure tables (in other words, fractions with denominator 10^7) are used in modern calculations. The Siddhantic assumption regarding the motions of the planets is that all of them travel the same number of Yojanas in their mean circular orbits during a Kalpa, and this constant is described as *Khakaksha* (खकक्षा) the circumference of the sphere to which solar rays extend. According to Bhaskara this constant is $187,120,692 \times 10^8$ while according to Suryasiddhanta it is $18,712,080,864 \times 10^6$. The inferences from this hypothesis just correspond to the findings of Newtonian astronomy, if we

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* The Hindus have apparently made several such lucky hits in many other branches of mathematics, as pointed out in the course of this paper.

† *Vide* p. 274, Appendix to the Translation of the *Suryasiddhanta* by the Rev. E. Burgess (1860).

assume the celestial bodies to revolve in concentric circles under a constant acceleration directed to the centre of the earth.

Side-tracked as they were by the wrong geocentric scent, the ancient Hindus like the Greeks reconciled the apparent discrepancies in the planetary motions by the contrivance of epicycles (नीचोच्चवृत्त) and eccentrics (मन्दवृत्त and शीघ्रवृत्त) and an elaborate calculus of approximations. One remarkable point about these contrivances is that they also lead, though by circuitous route more or less to the same results as modern methods to a first degree of approximation. Possibly they may be made to yield more refined results. The Indian physical explanation of the inequalities in the planetary motion may raise a smile among the moderns, but taken figuratively it merely means that there are certain centres of perturbation in the orbit which produce variations in velocity. Indian astronomers were fairly correct in their calculations of the moon's distance from the earth but erred in the distances of the sun and the planets as the latter were calculated from the hypothesis of constant velocity.

The other points of interest in Indian astronomy are the calculation of the moments of eclipses of the sun and the moon based on correct theory, the calculation of the length of the day by means of observations of the sun-dial, heliacal risings and settings, planetary conjunctions, parallax of the sun and the moon, the phases of the moon, the co-ordinates of certain well-known stars, etc.

Just as in the modern text-books of astronomy, in every siddhantic work there is a chapter on the description of instruments. But the description generally is so meagre and often abstruse that many western critics are tempted to believe that the instruments merely served to illustrate to the student the theory underlying astronomical calculations. The fact is that the explanations are purposely made scanty. The knowledge of the instruments was regarded as a privilege to be transmitted orally by the Guru with special care to the select few of his faithful disciples. This knowledge is given the exalted name of *Jyotishopanishad.*

There is enough evidence in the texts of the great Siddhanta writers to show that they actually verified the astronomical elements by personal observation of the heavenly bodies before incorporating them in their texts. Aryabhata explains briefly how he determined the mean motions of the planets by observing the intervals between their successive conjunctions with one another as well as with any particular star. Bhaskara expressly mentions that the instruments are meant for actual observation, and shrewdly remarks that more than instruments the mind behind them is valuable : धीरेकं पारमार्थिकं यन्त्रम् ।

Of the Hindu astronomers after Bhaskara, the following deserve notice :

(1) Mahendra Suri (*b.* 1320 A.D.) was the author of *Yantraraja*, a translation of a Persian work, mainly devoted to the construction, fitting and examination of instruments. The obliquity of the ecliptic is here given to be $23°35'$ and the rate of precession $45''$ a year. The work contains the latitudes and longitudes of 32 stars, and tables of sines and cosines for every degree.

(2) Gnanaraja (1503 A.D.) was the author of *Siddhanta Sundara.*

(3) Ganesa, a contemporary of Gnanaraja, was the author of numerous original astronomical works and commentaries. His *Grahalaghava*, supposed to be written in his thirteenth year, is a famous and popular book.

(4) Muniswara (*b.* 1603 A.D.) was a great scholar and a prolific writer on Astronomy and Astrology. His important work was *Siddhanta Sarva Bhauma.* He was an astrologer in the court of the Moghul Emperor Shah Jehan.

(5) Kamalakara (*b.* 1616 A.D.) was a contemporary and rival of Muniswara. He is well-known through the work, *Siddhanta tatva viveka*, based on the doctrines of *Surya-siddhanta.* He appears to be a good mathematical scholar versed also in Arabic works on Mathe-

matics and Astronomy. He makes frequent references to Mirza Ulugh Beg, a Persian royal astronomer of the tenth century.

(6) Jagannatha (*b.* 1652 A.D.) was a Telugu Brahmin. He was a great linguist versed in Sanskrit, Arabic, and Persian as well as a mathematician and astronomer. He was the chief pandit in the court of Jayasimha, king of Jaipur, and translated Ptolemy's ***Almagest*** from Arabic into Sanskrit. The translation entitled *Siddhanta Samrajya* contains also numerous citations from Ulugh Beg's work as well as from Jayasimha's original contributions. In addition it contains descriptions of various astronomical instruments set up by Jayasimha in the observatories at Benares and Ujjain. Jagannatha's ***Rekhaganita***, edited by Trivedi, is very well-known. It is the translation of Euclid's elements from Arabic into Sanskrit. Jagannatha's scholarship attracted the attention of the Emperor Aurangzeb, who appointed him chief pandit in his court.

No notable contributions appear to have been made to Hindu astronomy after the time of Bhaskara. Native astronomers had probably reached the highest level possible in the twelfth century with the amount of mathematical equipment of that age, for any further advance depended on advances in the auxiliary sciences of Mathematics, Optics, and Dynamics.

With the Muhammadan conquest of India, which began in the last quarter of the twelfth century, commenced the decay of Indian initiative in the positive sciences, and the little originality that was left after the fall of the Moghuls, was stifled by contact with Western civilization. Thus India has been passing through a gloom for the last six centuries, from which she is just recovering and attempting to make a new synthesis of her ancient civilization with the achievements of modern culture.